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EVERY COSEPARABLE GROUP MAY BE FREE

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Dedicated to the memory of Alan H. Mekler

ABSTRACT

We show that if $2^{\aleph_0}$ Cohen reals are added to the universe, then for every reduced non-free torsion-free abelian group A of cardinality less than the continuum, there is a prime p so that $\mathrm{Ext}_p(A, \mathbb{Z}) \neq 0$. In particular if it is consistent that there is a supercompact cardinal, then it is consistent (even with weak CH) that every coseparable group is free. The use of some large cardinal hypothesis is needed.

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1. Introduction

There are several approximations to being free for abelian groups. (From here on by **group** we will mean abelian group.) For some of these notions such as being hereditarily separable or being a Whitehead group it is well known that it is independent from the usual axioms of set theory whether or not every such group is free. (A group is **separable** if every finite set is contained in a free subgroup which is a direct summand and **hereditarily separable** if every subgroup is separable. A group A is a **Whitehead group** if $\mathrm{Ext}(A, \mathbb{Z}) = 0$. Any Whitehead group is hereditarily separable.) On the other hand it can be shown in ZFC that there are non-free groups of cardinality $\aleph_1$ which are $\aleph_1$-separable (i.e., any countable subset is contained in free countable direct summand). A somewhat mysterious class is the class of coseparable groups (to be defined later). Any hereditarily separable group is coseparable while every coseparable group is separable. So this class lies between those which exhibit independence phenomena and those for which we have absolute existence results.

Early on Chase [2] showed that CH implied the existence of a non-free coseparable group of cardinality $\aleph_1$. Later Sageev-Shelah [7] and Eklof-Huber [3] elaborated this result to construct non-free groups of cardinality $\aleph_1$ with specified p-rank of Ext (as p ranges over the primes). Their proofs continued to use CH. This reliance on CH was in itself unusual. In the cases where the existence of a group was known to be independent, the existence was also independent of CH. As well many results in abelian group theory that follow from CH can also be shown to follow from weak CH, i.e., $2^{\aleph_0} < 2^{\aleph_1}$. In this paper we will show that the use of CH is necessary. Namely we will show that is consistent with weak CH (assuming as always the consistency of ZFC) that every coseparable group of cardinality $\aleph_1$ is free. Furthermore if it is consistent that a supercompact cardinal exists then it is consistent (with weak CH if desired) that every coseparable group is free. A reference for the facts mentioned above and the ones we will quote below is [4].

A group is **$\aleph_1$-free** if every countable subgroup is free and, more generally, for an uncountable cardinal κ a group is **κ-free** if every subgroup of cardinality less than κ is free. A group A is **coseparable** if it is $\aleph_1$-free and every subgroup B such that A/B is finitely generated contains a direct summand C of A so that A/C is finitely generated. There are several equivalents to being coseparable. The one we will use throughout is that a group A is coseparable if and only if $\mathrm{Ext}(A, \mathbb{Z})$

is torsion-free or equivalently $\mathrm{Ext}_p(A,\mathbb{Z}) = 0$ for all primes p. Here $\mathrm{Ext}_p(A,\mathbb{Z})$ denotes the subgroup of $\mathrm{Ext}(A,\mathbb{Z})$ consisting of those elements whose order is a power of p. There is a useful characterization of when $\mathrm{Ext}_p(A,\mathbb{Z}) = 0$. For a group A and a prime p, $\mathrm{Ext}_p(A,\mathbb{Z}) = 0$ if and only if for every homomorphism $h: A \to \mathbb{Z}/p\mathbb{Z}$ there is a homomorphism $\hat{h}: A \to \mathbb{Z}$ (called a **lifting** of h) so that $\hat{h}/p = h$. (If f is a homomorphism to $\mathbb{Z}$, then f/p denotes the composition of f with the natural map to $\mathbb{Z}/p\mathbb{Z}$.)

Although (at least in some models of set theory) a subgroup of a coseparable group is not necessarily coseparable, every pure subgroup of a coseparable group is coseparable. The following proposition sums up the discussion.

PROPOSITION 1:

1. *If λ is the minimal cardinality of a non-free coseparable group then every coseparable group is λ-free.*
2. *Any homomorphism from a subgroup H of K to $\mathbb{Z}/p\mathbb{Z}$ can be extended to a homomorphism from K to $\mathbb{Z}/p\mathbb{Z}$ provided that H is p-pure in K, i.e., for any $a \in H$, $a \in pH$ if and only if $a \in pK$.*

2. Proofs

Most of our set theoretic notation will be standard. If κ is an infinite cardinal we will use $\mathrm{Fn}(I, J, \kappa)$ to denote the set of partial functions from I to J whose domain have cardinality less than κ. The poset $\mathrm{Fn}(\kappa, 2, \omega)$ is usually called the poset (or forcing) for adding κ Cohen reals. As is customary we will say that (at least) κ Cohen reals are added to a set theoretic universe if we force with $\mathrm{Fn}(\kappa, 2, \omega)$ (or $\mathrm{Fn}(\mu, 2, \omega)$ for some $\mu \geq \kappa$) The principal lemma we will use is the following.

LEMMA 2: *Suppose κ is a cardinal and $\mathbb{P} = \mathrm{Fn}(\kappa, 2, \omega)$. Then $\mathbb{P}$ forces that every coseparable group of cardinality less than κ is free.*

This lemma has as immediate consequences our main theorems.

THEOREM 3: *It is consistent with both $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} = 2^{\aleph_1}$, that every coseparable group of cardinality $\aleph_1$ is free.*

THEOREM 4: *If it is consistent that a supercompact cardinal exists then it is consistent with both $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} = 2^{\aleph_1}$, that every coseparable group is free.*

Proof: Shelah (see [1]) has shown that if κ is a supercompact cardinal and κ Cohen reals are added to the universe then every κ-free group is free. So after adding κ Cohen reals we have, by Lemma 2 and Proposition 1, that every coseparable group is κ-free and hence free. As well we have that $2^{\aleph_0} = \kappa = 2^{\aleph_1}$.

For the other part of the theorem, it suffices to know that the consistency of a supercompact cardinal implies the consistency of a cardinal κ such that $\kappa < 2^{\aleph_1}$ and if κ Cohen reals are added to the universe then every κ-free group is free. This result is proved in Corollary 20. ∎

We will also show that the use of some large cardinal is necessary. (The consistency of the statement "every coseparable group is free" implies the consistency of the existence of measurable cardinals.) Also we will show by a more complicated forcing that assuming the consistency of a supercompact cardinal it is consistent that every coseparable group is free and the continuum is as small as possible, i.e., $\aleph_2$.

There are various cases to consider in the proof of the main lemma. At various times we will have to consider the case of a free group F and its p-adic completion. Fix a set of free generators X for F. Then any element y of the p-adic completion can be written uniquely as $\sum p^n y_n$ where each y_n is a linear combination of elements of X such that all the coefficients are between 0 and $p-1$ (inclusive). If we have fixed a set of free generators then given y, y_n will always denote the element above. First we prove two easy and useful propositions.

PROPOSITION 5: *Suppose A is contained in the p-adic completion of a free group F and X is a set of free generators for F. Assume $h \in \mathrm{Hom}(A, \mathbb{Z}/p\mathbb{Z})$ and $\hat{h}$ is a lifting to $\mathbb{Z}$. If $x, y \in A$, $\hat{h}(x) = \hat{h}(y)$ and n is the least element so that $x_n \neq y_n$ then $h(x_n) = h(y_n)$.*

Proof: By subtracting $\sum_{i<n} p^i x_i$ from each side and then dividing by p^n, we have $\hat{h}(\sum_{n\leq m<\infty} p^{m-n} x_m) = \hat{h}(\sum_{n\leq m<\infty} p^{m-n} y_m)$. So $\hat{h}(x_n) \equiv \hat{h}(y_n) \bmod p$ ∎

PROPOSITION 6: *Let A, F and X be as above. Assume that there is a finite set Y of elements which are not divisible by p and a set $B \subseteq A$ so that $|B| > |F|$ and for all $x \neq y \in B$, $x_n - y_n \in Y$, where n is the least natural number so that $x_n \neq y_n$. Then $\mathrm{Ext}_p(A, \mathbb{Z}) \neq 0$.*

Proof: Suppose not. For each $a \in Y$, choose h_a so that $h_a(a) \neq 0$. Let $\hat{h}_a$ be a lifting of h_a. Choose $B' \subseteq B$ so that $|B'| > |F|$ and for all $x, y \in B'$ and $a \in Y$, $\hat{h}_a(x) = \hat{h}_a(y)$. By the last proposition for $x \neq y \in B'$ and $a \in Y$, $x_n - y_n \neq a$, where n is the least natural number so that $x_n \neq y_n$. This contradicts the choice of Y. ∎

We will have occasion to use the following *ad hoc* definition. If A is an abelian group a function $h: A \to \mathbb{Z}/p\mathbb{Z}$ is a *counterexample to the vanishing of* $\mathrm{Ext}_p(A, \mathbb{Z})$ if h has no lifting to a map to $\mathbb{Z}$.

THEOREM 7: *Suppose A is contained in the p-adic completion of a free group F and $|A| > |F|$. Then after adding at least $|F|$ Cohen reals to the universe* $\mathrm{Ext}_p(A, \mathbb{Z}) \neq 0$.

Proof: By splitting the forcing into two parts (and adding $|F|$ Cohen reals at the second step) we can assume that exactly $|F|$ Cohen reals are added. Let X be a set of free generators of F and let h be a Cohen generic function from X to p. More exactly, we can assume the Cohen reals are added by forcing with $\mathrm{Fn}(X, p, \omega)$. Let h be a generic set for this forcing and we will also let h denote the unique extension to a homomorphism from A to $\mathbb{Z}/p\mathbb{Z}$. (There are many other schemes to produce a generic function.) Suppose $\hat{h}$ is forced (by some condition) to be a lifting of h. For each $a \in A$, choose a condition q_a and an integer m_a so that $q_a \Vdash \hat{h}(a) = m_a$. Since $|A| > \max\{|X|, |\mathbb{Z}|\}$, there is a condition q so that for some m and some B of cardinality greater than $|F|$, $q \Vdash \hat{h}(x) = m$, for all $x \in B$. Let X_0 be the domain of q and let Y be the set of linear combinations of elements of X_0 whose coefficients are between $-p+1$ and $p-1$.

Suppose first that for all $x, y \in B$, $x_n - y_n \in Y$ where n is the least natural number such that $x_n \neq y_n$. Then we are done by Proposition 6. So we can choose $x, y \in B$ so that this is not the case. Choose now q_1 extending q so that $q_1 \Vdash h(x_n - y_n) \neq 0$. This contradicts Proposition 5. So no such lifting exists. ∎

The non-existence of a group A which is coseparable and satisfies the hypotheses of the theorem is not a consequence of ZFC. It is consistent that there is a Whitehead group of cardinality $\aleph_1$ which is contained in the 2-adic completion of a countable set [9]. In fact the non-free coseparable group constructed by Chase [2], assuming CH, is contained in the $\mathbb{Z}$-adic completion of a countable set.

We can strengthen the previous theorem to allow us to calculate the p-rank.

THEOREM 8: *Suppose A is contained in the p-adic completion of a free group F and $|A| > |F|$. Then if $\lambda \geq |F|$ and λ Cohen reals are added to the universe, $|\mathrm{Ext}_p(A, \mathbb{Z})| \geq \lambda$.*

Proof: There are two cases to consider. First assume that we have a collection $\{Y_n : n < \omega\}$ of disjoint finite subsets of F so that for all n, no element of Y_n is divisible by p modulo the subgroup generated by $\bigcup_{m \neq n} Y_m$, the subgroup generated by $\bigcup_{n<\omega} Y_n$ is p-pure in F and for all n there is a set $B \subseteq A$ of cardinality greater than the cardinality of F so that if $x, y \in B$ then $x_i - y_i \in Y_n$ where i is the least natural number so that $x_i \neq y_i$. In this case by (the proof of) Proposition 6 we can find elements $a_n \in Y_n$ such that if $h: A \to \mathbb{Z}/p\mathbb{Z}$ is a homomorphism which is non-zero on a_n then h does not lift. There are $2^{\aleph_0}$ homomorphisms $\{h_i : A \to \mathbb{Z}/p\mathbb{Z}: \quad i < 2^{\aleph_0}\}$ such that for all $i \neq j$ there is n such that $(h_i - h_j)(a_n) \neq 0$. So for any $i \neq j$, $h_i - h_j$ does not lift to a map to $\mathbb{Z}$ and hence does not represent 0 in $\mathrm{Ext}_p(A, \mathbb{Z})$. As h_i and h_j represent different elements of $\mathrm{Ext}_p(A, \mathbb{Z})$, $|\mathrm{Ext}_p(A, \mathbb{Z})| = 2^{\aleph_0}$.

For the second case we assume that the first does not hold. In this case we can find a pure finitely generated subgroup Y of F so that for all finite subsets Z of F if every element of Z is not divisible by p modulo Y then there is no set B of cardinality greater than $|F|$ so that for all $x, y \in B$, $x_n - y_n \in Z$ where n is the least natural number so that $x_n \neq y_n$. We consider the Cohen reals as giving λ generic functions $\{h_\alpha : \alpha < \lambda\}$ from F to $\mathbb{Z}/p\mathbb{Z}$ which are 0 on Y. In this case we can repeat the second part of the argument in Theorem 7, since for all $\alpha \neq \beta$, $h_\alpha - h_\beta$ is generic. ∎

We now turn to the proof of the main Lemma. Since if A is a non-free coseparable group of minimal cardinality, A is $|A|$-free, it is enough to show the result for all λ-free groups A of cardinality $\lambda < 2^{\aleph_0}$. By Shelah's singular compactness theorem [8], λ is regular. We will consider a λ-filtration of A (i.e., write A as a union of a continuous chain $(A_\alpha : \alpha < \lambda)$ of pure subgroups of cardinality less than λ such that for all α, if A/A_α is not λ-free then $A_{\alpha+1}/A_\alpha$ is not free). There will be two cases to consider. The first and easier case is dealt with by the following Theorem.

THEOREM 9: *Suppose that $|A| = \lambda$ and A is λ-free. Also suppose that $(A_\alpha : \alpha < \lambda)$ is a λ-filtration of A and for a stationary set E, after adding at least λ Cohen reals, $\mathrm{Ext}_p(A_{\alpha+1}/A_\alpha, \mathbb{Z}) \neq 0$ for all $\alpha \in E$. Furthermore assume that for all*

$\alpha \in E$ the non-vanishing of $\mathrm{Ext}_p(A_{\alpha+1}/A_\alpha, \mathbb{Z})$ *is witnessed by a function in the ground model. Then if we add at least λ Cohen reals,* $\mathrm{Ext}_p(A, \mathbb{Z})$ *has rank 2^λ.*

Also if weak diamond of E holds then $\mathrm{Ext}_p(A, \mathbb{Z})$ *has rank 2^λ, provided that for all $\alpha \in E$,* $\mathrm{Ext}_p(A_{\alpha+1}/A_\alpha, \mathbb{Z}) \neq 0$.

Proof: This is a fairly routine weak diamond proof. In the case where we add Cohen reals the proof can be done using the definable weak diamond (see [5]). In order to pursue the main case we will give the proof using the Cohen reals directly. As well we will only show that $\mathrm{Ext}_p(A, \mathbb{Z}) \neq 0$. (The proof that $\mathrm{Ext}_p(A, \mathbb{Z})$ has rank 2^λ is similar.) Choose a basis $X = \cup_{\alpha<\lambda} X_\alpha$ of A/pA so that (under the obvious identifications) X_α is a basis of $(A_{\alpha+1}/A_\alpha)/p(A_{\alpha+1}/A_\alpha)$. For $\alpha \in E$, let $h_{1\alpha}$ be a map, in the ground model, from X_α to $\mathbb{Z}/p\mathbb{Z}$ which does not lift to a homomorphism from $A_{\alpha+1}/A_\alpha$ to $\mathbb{Z}$ (even in the forcing extension). Let $h_{0\alpha}$ denote the homomorphism which is constantly 0 on X_α. The important point to notice here is that if f is any homomorphism from A_α to $\mathbb{Z}$ then at most one of $f/p + h_{0\alpha}$ and $f/p + h_{1\alpha}$ lifts to a homomorphism of $A_{\alpha+1}$ which agrees with f on A_α. Otherwise if f_0, f_1 were two such liftings then $f_1 - f_0$ would contradict the choice of $h_{1\alpha}$. For $\alpha \notin E$, let $h_{0\alpha}$ and $h_{1\alpha}$ be any functions from X_α to $\mathbb{Z}/p\mathbb{Z}$.

Now we can use λ Cohen reals to define a map which cannot be lifted. We can assume that the forcing is $\mathrm{Fn}(\lambda, 2, \omega)$ and G is a generic function from λ to 2. Let h from X to $\mathbb{Z}/p\mathbb{Z}$ be $\sum_{\alpha<\lambda} h_{G(\alpha)\alpha}$. Suppose $\hat{h}$ is forced to be a lifting of h (without loss of generality we can assume that it is forced to be a lifting by the empty condition). Since Cohen forcing is c.c.c. we can find $\alpha \in E$ so that $\hat{h}\restriction A_\alpha$ is decided by $G\restriction\alpha$. So there is $l = 0, 1$ and a condition, q, whose domain is contained in α such that q forces that $\hat{h}\restriction A_\alpha$ does not extend to a function which lifts $h\restriction A_\alpha + h_{l\alpha}$. If we choose r so that r extends q and $r(\alpha) = l$ then r forces that $\hat{h}$ is not a lifting of h. This is a contradiction. ∎

The last case of the main lemma is the following.

THEOREM 10: *Suppose that λ is a regular cardinal and A is a group of cardinality λ so that A is λ-free and there is a λ-filtration of A so that for a stationary set E and all $\alpha \in E$ there is $a \in A_{\alpha+1} \setminus A_\alpha$ so that a is in the p-adic completion of A_α. Then if we add at least λ Cohen reals to the universe,* $\mathrm{Ext}_p(A, \mathbb{Z}) \neq 0$. *In fact,* $|\mathrm{Ext}_p(A, \mathbb{Z})| \geq \lambda$.

Proof: In view of the Theorem 7 (or Theorem 8 for the stronger result) we can assume that the p-adic closure in A of any set of cardinality $< \lambda$ has size $< \lambda$.

Applying Fodor's lemma we get that all but a non-stationary subset of E elements of E are of cofinality ω. (Otherwise there would be some α so that the p-adic closure in A of A_α has cardinality λ.) Also by Theorem 7, we can choose the filtration so that for all α the p-adic closure in A of A_α is contained in $A_{\alpha+1}$.

Fix a basis X for A/pA. We now redefine the filtration of A. We can assume A is the union of a λ-filtration where $A_\alpha = N_\alpha \cap A$ where $(N_\alpha\colon \alpha < \lambda)$ is an increasing sequence of elementary submodels of cardinality less than λ of some $(\mathrm{H}(\kappa), \in)$ with the usual good properties (including $X \in N_0$). Since the new filtration agrees with the old one on a closed unbounded set we can assume E consists only of ordinals of cofinality ω.

Since the p-adic closure in A of any set of cardinality less than λ has cardinality less than λ, X has cardinality λ. We can assume that the forcing is $\mathrm{Fn}(X, p, \omega)$. Let h be the generic function from X to $\mathbb{Z}/p\mathbb{Z}$. Note that the obvious name for h is in N_0. Suppose that $\hat{h}$ is a lifting of h.

Choose a limit ordinal $\delta \in E$ and let a be an element of $A_{\delta+1} \setminus A_\delta$ which is in the p-adic closure in A of A_δ. Choose q which determines $\hat{h}(a)$, say it forces the value to be m. Choose $\alpha < \delta$ so that $q \cap N_\delta = q \cap N_\alpha$. Call this restriction q_1. Next take n maximal so that a is p^n-divisible modulo A_α. (Since the p-adic closure in A of A_α is contained in A_δ such an n must exist.) Choose $y \in A_\alpha$ such that $p^n \mid a - y$. Let $q_2 \in N_\alpha$ be an extension of q_1 which determines $\hat{h}(y)$. Now choose $b \in A_\delta$ so that $p^{n+1} \mid a - y - b$. Notice that p^n is the exact power of p which divides b (even modulo A_α). Since $p^{n+1} \nmid b + A_\alpha$, if we write b/p^n as a (infinite) sum of elements of X there will be elements whose coefficients are not divisible by p which lie outside A_α. Hence there is a condition $q_3 \in N_\delta$ extending q_2 so that q_3 forces $h(b/p^n) \not\equiv (m - \hat{h}(y)/p^n) \bmod p$. Let r be a common extension of q_3 and q. This gives a contradiction.

To get the stronger statement on the cardinality of $\mathrm{Ext}_p(A, \mathbb{Z})$ it is enough as we did in Theorem 8 to use the Cohen reals to code λ generic functions and then note that for any two of them their difference is also generic. ∎

Finally we can complete the proof of the main lemma.

Proof of Lemma 2: The proof is by induction on $\lambda < \kappa$. We show that any coseparable group A of cardinality λ is free. Suppose that A is a coseparable group, $|A| = \lambda < \kappa$ and that the induction hypothesis is true for all $\rho < \lambda$. Assume, for the sake of contradiction that A is not free. By the induction hypothesis, A is

λ-free and by the singular compactness theorem [8] λ is regular. Let $(A_\alpha\colon \alpha < \lambda)$ be a λ-filtration of A s.t. A_α pure subgroup $E^* \stackrel{\text{df}}{=} \{\alpha : A_{\alpha+1}/A_\alpha \text{ not free}\}$ is stationary. There are two main cases: either for a stationary set $E \subseteq \lambda$, $A_{\alpha+1}/A_\alpha$ contains a non-zero divisible group or not. In the first case we have by Theorem 10 that for every p there is a counterexample to the vanishing of $\mathrm{Ext}_p(A, \mathbb{Z})$.

The second case divides into two subcases. If there is a stationary set E so that for all $\alpha \in E$, $A_{\alpha+1}/A_\alpha$ is $\aleph_1$-free not free then by the induction hypothesis and the fact that there are only countably many primes we can choose a stationary subset S of E and a prime p so that for all $\alpha \in S$ there is a counterexample to the vanishing of $\mathrm{Ext}_p(A_{\alpha+1}/A_\alpha, \mathbb{Z})$. By absorbing at most λ of the Cohen reals into the ground model we can assume that A, $(A_\alpha\colon \alpha < \lambda)$, E and the counterexamples to the vanishing of $\mathrm{Ext}_p(A_{\alpha+1}/A_\alpha, \mathbb{Z})$ are all in the ground model. In this case we can apply Theorem 9.

In the final subcase there is a stationary set E so that for all $\alpha \in E$, $A_{\alpha+1}/A_\alpha$ is not $\aleph_1$-free and contains no non-zero divisible subgroup. So for all $\alpha \in E$ there is a prime p_α so that there is a counterexample to the vanishing of $\mathrm{Ext}_{p_\alpha}(A_{\alpha+1}/A_\alpha, \mathbb{Z})$ (see [4] XII.2.7 for a proof of this standard fact). If we choose a prime p and S a stationary subset of E so that $p_\alpha = p$ for all $\alpha \in S$, we can again apply Theorem 9 (as above). ∎

Since we have been able to calculate the p-ranks of our groups in all cases we have the following theorem on the structure of Ext.

THEOREM 11: *Suppose it is consistent that a supercompact cardinal exists. Then it is consistent with either* $2^{\aleph_0} = 2^{\aleph_1}$ *or* $2^{\aleph_0} < 2^{\aleph_1}$ *that for any group* A *either* $\mathrm{Ext}(A, \mathbb{Z})$ *is finite or* $\mathrm{Ext}(A, \mathbb{Z})$ *has rank at least* $2^{\aleph_0}$.

It is natural to ask whether the use of the large cardinals is necessary. To show that it is, we will give a construction of a coseparable group (which is similar to Chase's) from principles whose negation implies consistency of the existence of many measurable cardinals. By the principle, $*(\lambda, \aleph_0)$, we will mean that there exists a family $\{S_i\colon i < \lambda^+\}$ of countable subsets of λ such that for any collection $I \subseteq \lambda^+$ of size λ there exists $\{S_i^*\colon i \in I\}$ where for all $i \in I$ $S_i \setminus S_i^*$ is finite and for all $i \neq j$, $S_i^* \cap S_j^* = \emptyset$. (See page 157 of [4] for details.)

THEOREM 12: *Suppose that* λ *is a cardinal,* $\mathrm{cf}(\lambda) = \omega$, $2^\lambda = \lambda^+$, $*(\lambda, \aleph_0)$ *holds and for all* $\mu < \lambda$, $2^\mu < \lambda$. *Then there is a non-free coseparable group of* *cardinality* λ^+.

Proof: Let $\{S_i\colon i < \lambda^+\}$ be as guaranteed by $*(\lambda, \aleph_0)$ and for each i let $\{s^i_n\colon n < \omega\}$ be an enumeration of S_i. Let G_0 be the group freely generated by λ ($= \bigcup_{i<\lambda^+} S_i$). Let $\{f_i\colon i < \lambda^+\}$ enumerate the homomorphisms from G_0 to $\mathbb{Z}/p\mathbb{Z}$ as p varies over the primes. We will inductively define a strictly increasing chain of free groups G_α and homomorphisms g_α from $G_{\alpha+1} \to \mathbb{Z}$ for $\alpha < \lambda^+$ so that g_α is a lifting of f_α, G_α/G_0 is divisible and for all $\beta < \alpha$, g_β extends to a homomorphism from G_α to $\mathbb{Z}$. (Since G_α/G_β is divisible there is at most one extension.) If we can do this then $G = \bigcup_{\alpha<\lambda^+} G_\alpha$ is the group required. G is not free. Since G/G_0 is divisible, for every prime p every homomorphism from G to $\mathbb{Z}/p\mathbb{Z}$ is uniquely determined by its restriction to G_0. By the construction every homomorphism from G_0 to $\mathbb{Z}/p\mathbb{Z}$ lifts, so G will be as required.

It remains to do the construction. We will work inside the $\mathbb{Z}$-adic completion of G_0 so our use of infinite sums will be justified. Choose a function $h\colon \lambda \to \lambda$ so that the inverse image of any ν has cardinality λ. For each α we will choose $x^\alpha_{n,0} \neq x^\alpha_{n,1}$ so that $h(x^\alpha_{n,l}) = s^\alpha_n$ and define $x_\alpha = \sum_{n<\omega} n!(x^\alpha_{n,0} - x^\alpha_{n,1})$. The group $G_{\alpha+1}$ will be the pure closure of $G_\alpha \cup \{x_\alpha\}$. At limit ordinals we will take unions. If we do this then for each α the group G_α will be free but the group G will not be free (see [6] or [4] Theorem VII.2.13).

Now we make the choices. Suppose we are at stage α. We have already chosen $\{g_\beta\colon \beta < \alpha\}$ a set of at most λ functions. Write $\alpha = \bigcup_{n<\omega} w_n$, where the cardinality of each w_n is less than λ and $w_n \subseteq w_{n+1}$. Since $2^{|w_n|} < \lambda$, for each n we can choose $x^\alpha_{n,0} \neq x^\alpha_{n,1}$ so that $h(x^\alpha_{n,l}) = s^\alpha_n$ and for all $\beta \in w_n$, $g_\beta(x^\alpha_{n,0}) = g_\beta(x^\alpha_{n,1})$. Then the homomorphisms extend to the infinite sum (namely x_α) since each homomorphism is 0 on all but a finite number of the terms being summed. Finally we must choose g_α. Since $G_{\alpha+1}$ is free there is some lifting of f_α to $G_{\alpha+1}$, so we can choose g_α as required. ∎

As we have mentioned the statement that the hypothesis of the theorem fails at all cardinals implies the consistency of many large cardinals. So the use of some large cardinal hypothesis was necessary. One might also want to know if the continuum needs to be as large as we have made it. The answer to this question is no. Assuming the consistency of the existence of a supercompact cardinal it is possible to show that it is consistent with $2^\lambda = \lambda^+ + \aleph_2$ that every coseparable group is free. To prove this result we will divide our effort between a set-theoretic statement and a group theoretic one. First we need to define a forcing axiom. By $\mathrm{Ax}^+(\aleph_1\text{-complete forcing})$ we mean the statement "if $\mathbb{Q}$ is an $\aleph_1$-complete forcing and $\tilde{S}$ is a $\mathbb{Q}$-name and $\Vdash_{\mathbb{Q}}$ "$\tilde{S}$ is a stationary subset of ω_1" then for some directed $G \subseteq \mathbb{Q}$ the set $\{\alpha < \omega_1\colon \text{there is } q \in G,\ q \Vdash \alpha \in \tilde{S}\}$ is stationary. "

The point of $\mathrm{Ax}^+(\aleph_1\text{-complete forcing})$ is to give a reflection principle which shows that certain groups are not $\aleph_2$-free.

LEMMA 13: *Suppose that $\mathrm{Ax}^+(\aleph_1$-complete forcing) holds and $\mathbb{P}$ is an $\aleph_2$-complete forcing. (By $\aleph_2$-complete we mean that any directed subset of cardinality $\aleph_1$ has an upper bound.) If G is $\mathbb{P}$-generic then the following statement is true in the generic extension.*

> *Suppose λ is a regular cardinal, A is a group of cardinality λ and $\{A_\alpha\colon \alpha < \lambda\}$ is a λ-filtration of A such that*
>
> $$E = \{\alpha\colon A_{\alpha+1}/A_\alpha \text{ is not } \aleph_1\text{-free}\}$$
>
> *is stationary. Then A is not $\aleph_2$-free.*

Proof: Without loss of generality we can assume that the set underlying A is λ and $\tilde{A}$ is a $\mathbb{P}$-name for the group structure on λ which is forced by the empty condition to be as in the hypothesis. We will show that there is a condition which forces that $\tilde{A}$ is not $\aleph_2$-free. Let $\mathbb{Q}$ be the product of $\mathbb{P}$ with the forcing $\mathbb{R}$ which adds a generic function $\tilde{f}$ from ω_1 onto λ by countable conditions. Let $\tilde{S}$ be the $\mathbb{Q}$-name of a subset of ω_1 such that $q \Vdash \alpha \in \tilde{S}$ if and only if for some $\beta > \alpha$, q determines $\tilde{f}\restriction\beta$ and the following holds. Let X be the determined value of the image of $\tilde{f}\restriction\beta$. Then q determines the group structure of $\tilde{A}$ on X and if Y is the determined image of $\tilde{f}\restriction\alpha$ then X and Y are given group structures by q and X/Y contains a non-free group of finite rank.

We need to see that $\tilde{S}$ is forced to be stationary. First recall that for any groups C and D, where $C \subseteq D$, D/C is not $\aleph_1$-free if and only if there is a non-free subgroup of D/C of finite rank. Let $\tilde{T}$ be the $\mathbb{R}$-name for $\tilde{S}^{G\times\tilde{H}}$, where $\tilde{H}$ is the canonical name for an $\mathbb{R}$-generic set. It is enough to see that in $V[G]$, $\tilde{T}$ is forced to be stationary. This task is quite easy. Let $\tilde{C}$ be an $\mathbb{R}$-name in $V[G]$ for a closed unbounded subset of ω_1. By taking a new filtration of A we can assume that E consists entirely of ordinals of cofinality ω. Let r be an element of $\mathbb{R}$, i.e., r is a function from some countable ordinal μ to λ. Choose N an elementary submodel of some appropriate $(\mathrm{H}(\kappa), \in)$ so that $|N| < \lambda$ and $N \cap A = A_\alpha$ for some $\alpha \in E$ also we can assume that r and everything else we have been talking about are elements of N. Choose a countable subgroup X_0 of $A_{\alpha+1}$ so that $X_0 + A_\alpha/A_\alpha$ is not free. It is standard to choose an increasing sequence $r = r_0, r_1, \ldots$ of conditions in $N \cap \mathbb{R}$ where the domain of r_n is μ_n so that there is an increasing sequence $(\nu_n : n < \omega)$ of ordinals such that for all n, $r_n \Vdash \nu_n \in \tilde{C}$, $\mu_n < \nu_{n+1}$ and $\bigcup_{n<\omega} \mathrm{rge}(r_n)$ is a subgroup of A_α which contains $X_0 \cap A_\alpha$. Let r_ω denote $\bigcup_{n<\omega} r_n$. Finally let $r_{\omega+1}$ be an extension of r_ω so that the range of $r_{\omega+1}$ is $X_0 + \mathrm{rge}(r_\omega)$. Then $r_{\omega+1}$ forces that $\bigcup_{n<\omega} \nu_n \in \tilde{C} \cap \tilde{T}$.

So by $\mathrm{Ax}^+(\aleph_1$-complete forcing) there is a directed set H of cardinality $\aleph_1$ such that $\{\alpha$: there is $q \in H, q \Vdash \alpha \in \tilde{S}\}$ is stationary. Let p be an upper bound to the first coordinates of $\{q : q \in H\}$. By the definition of $\tilde{S}$, H determines a function, f from ω_1 to λ and a group structure on the range of f which p forces to coincide with a subgroup of $\tilde{A}$. Further the definition of $\tilde{S}$ and the choice of H guarantees that the group structure on the range of f is not free. ∎

LEMMA 14: *Suppose that λ is a regular cardinal, A is not coseparable and $|A| = \lambda$. If $\mathbb{Q}$ is a λ-complete notion of forcing then $\mathbb{Q} \Vdash A$ is not coseparable.*

Proof: Without loss of generality we can assume that the set underlying A is λ. Suppose p is a prime and $h: A \to \mathbb{Z}/p\mathbb{Z}$ is a function in the ground model which does not lift. Suppose that $\tilde{g}$ is forced to be a lifting of h. Choose an increasing sequence of conditions $(q_\alpha : \alpha < \lambda)$ so that q_α determines $\tilde{g}(\alpha)$. Then f defined by $f(\alpha) = n$ if $q_\alpha \Vdash \tilde{g}(\alpha) = n$ is a lifting of h. ∎

THEOREM 15: *Suppose that the following statements are true: for all infinite cardinals λ, $2^\lambda = \lambda^+ + \aleph_2$; every coseparable group of power less than $2^{\aleph_0}$ is free; and $\mathrm{Ax}^+(\aleph_1$-complete forcing). Then there is a generic extension such that*

(1) *cardinalities and cofinalities are preserved,*
(2) *for all infinite cardinals λ, $2^\lambda = \lambda^+ + \aleph_2$,*
(3) *every coseparable group is free,*
(4) *suppose λ is a regular cardinal, A is a group of cardinality λ and $\{A_\alpha : \alpha < \lambda\}$ is a λ-filtration of A such that $E = \{\alpha$: $A_{\alpha+1}/A_\alpha$ is not $\aleph_1$-free$\}$ is stationary. Then A is not $\aleph_2$-free.*

Proof: The forcing is an iteration over all regular cardinals greater than or equal $\aleph_2$, where the support is an initial segment. The forcing $\tilde{Q}_\lambda$ is the $\mathbb{P}_\lambda$-name for adding a Cohen subset of λ. The result of this forcing we will use is that $\diamondsuit(S)$ holds for all regular $\lambda > \aleph_1$ and S a stationary subset of λ. Properties (1) and (2) are standard. Property (4) has been proved in Lemma 13. It remains to verify that every coseparable group is free. Since the forcing is ω_2-complete, we have, by Lemma 14, that every coseparable group of cardinality at most $\aleph_1$ is free. Suppose that A is a non-free coseparable group of cardinality λ where λ is minimal. (Note that $\lambda > \aleph_1$.) We can assume that the set underlying A is λ. By forcing with the initial segment which adds the subsets to μ for $\mu \leq \lambda$, we can assume we are working in a universe satisfying (1), (2), (4) where every coseparable group of cardinality less than λ is free, A is a non-free λ-free group and $\diamondsuit(S)$ holds for every stationary subset of λ. Choose a λ-filtration $(A_\alpha\colon \alpha < \lambda)$ of A. The filtration should be chosen so that $A_{\alpha+1}/A_\alpha$ is not free if A/A_α is not λ-free.

Let $E = \{\alpha\colon A_{\alpha+1}/A_\alpha \text{ is not } \aleph_1\text{-free}\}$. Then E is not stationary, since by (4), if E were stationary A would not be $\aleph_2$-free. By the inductive hypothesis, $\{\alpha\colon \mathrm{Ext}_p(A_{\alpha+1}/A_\alpha, \mathbb{Z}) \neq 0\}$ is stationary for some p. (In fact, for all p.) Hence by Theorem 9 and the fact that $\diamondsuit(S)$ holds for all stationary subsets of λ we are done. ∎

It remains to show the hypothesis is consistent.

THEOREM 16: *Suppose it is consistent that a supercompact cardinal exists then it is consistent that: for all infinite cardinals λ, $2^\lambda = \lambda^+ + \aleph_2$; every coseparable group of power less than $2^{\aleph_0}$ is free; and* $\mathrm{Ax}^+(\aleph_1$*-complete forcing*).

Proof: Since for any $\aleph_1$-complete forcing $\mathbb{P}$, there is some cardinal λ so that $\mathbb{P} \times \mathrm{Fn}(\omega_1, \lambda, \omega_1)$ is equivalent to $\mathrm{Fn}(\omega_1, \lambda, \omega_1)$ it is enough to consider posets of the form $\mathrm{Fn}(\omega_1, \lambda, \omega_1)$ in the verification of $\mathrm{Ax}^+(\aleph_1$-complete forcing). We will call $\mathrm{Fn}(\omega_1, \lambda, \omega_1)$ the forcing for collapsing λ to ω_1. Suppose that κ is supercompact. We can assume that GCH holds in the ground model. Let $f : \kappa \to \kappa$ be a function such that for any μ and λ there is some M and an elementary embedding of $j\colon V \to M$ so that $j(f)(\kappa) = \mu$ and ${}^\lambda M \subseteq M$. The forcing is an iteration $(\mathbb{P}_i, \tilde{Q}_i\colon i < \kappa)$, where if $i = 2j$, $\tilde{Q}_i$ is the $\mathbb{P}_i$-name for the forcing collapsing $f(j)$ to ω_1 and for $i = 2j+1$, $\tilde{Q}_i$ is the forcing for adding $\aleph_1$ Cohen reals. A function p is in $\mathbb{P}_\kappa$ if for all i, $p\restriction i \Vdash p(i) \in \tilde{Q}_i$, the support of p intersect the even ordinals is countable and the support of p intersect the odd ordinals is finite.

It is standard to verify that $\mathrm{Ax}^+(\aleph_1$-complete forcing) is forced to hold. As well κ is forced to be either $\aleph_1$ or $\aleph_2$. We will show that $\aleph_1$ is preserved by this forcing and that every coseparable group of cardinality ω_1 is free in the forcing extension. (Notice that the first statement is implied by the second since otherwise CH would hold and so there would be a non-free coseparable group of cardinality $\aleph_1$. In fact we will verify that ω_1 is preserved as part of the verification that all coseparable groups of cardinality $\aleph_1$ are free.) We will only sketch the proof and leave details to the reader. We can reconstruct the iteration differently. For any i, let $\mathbb{R}_i$ be the finite support iteration of $(\tilde{Q}_{2j+1}\colon 2j+1<i)$. The forcing $\mathbb{P}_\kappa$ can be rewritten as $\mathbb{R}_\kappa * \tilde{S}_\kappa$, where $\tilde{S}_i$ is the countable support iteration of $(\tilde{Q}_{2j}\colon 2j<i)$. We claim that if H is an $\mathbb{R}_\kappa$-generic set and G is $\mathbb{P}_\kappa$-generic, $G\cap\mathbb{R}_\kappa = H$ then $V[H]$ and $V[G]$ have the same countable sets of ordinals. To show this we show by induction on i, that $V[H_i]$ and $V[G_i]$ have the same countable sets of ordinals. Here H_i (respectively, G_i) denotes the restriction of H (respectively, G) to $\mathbb{R}_i$ (respectively, $\mathbb{P}_i$). In particular we have from the induction hypothesis that for any λ, $\mathrm{Fn}(\omega_1,\lambda,\omega_1)^{V[H_i]} = \mathrm{Fn}(\omega_1,\lambda,\omega_1)^{V[G_i]}$. So we can in the forcing replace $\tilde{Q}_{2j}$ by the $\mathbb{R}_{2j}$-name for the forcing for collapsing $f(j)$ to ω_1, call this name $\tilde{Q}'_{2j}$.

Any element of ${\tilde{Q}'_{2j}}{}^{H_{2j}}$ can be of the form $\{\{(\check{\alpha},\check{\beta})\}\times A_{\alpha\beta}\colon \alpha<\gamma,\beta<f(j)\}$ where $\gamma<\omega_1$, for all α and β $A_{\alpha\beta}\subseteq\mathbb{R}_{2j}$, for all α,β,τ, if $p\in A_{\alpha\beta}$ and $q\in A_{\alpha\tau}$ then p and q are incompatible, and for all α, $\bigcup\beta<f(j)A_{\alpha\beta}$ contains a maximal antichain. Call such names *pleasant*. Notice that any pleasant name is forced by the empty condition to be a function with countable domain from ω_1 to λ. As well, if q is any name which is forced by the empty condition to be such a function then there is a pleasant name which the empty condition forces to extend q. The important property that we will use is that the union of a countable chain of pleasant names which is forced by the empty condition to be increasing in stength is again a pleasant name. With this observation in hand, we can prove the claim. There are two cases to consider, the successor case and the limit case. We will just do the limit case as the successor one is similar. Suppose i is a limit ordinal and $\tilde{g}$ is a $\mathbb{P}_i$-name for a function from ω to the ordinals. It is enough to show that there is a name $\tilde{s}$ in $\tilde{S}_i$ so that the empty condition in $\mathbb{R}_i$ forces that $\tilde{s}$ determines the value of $\tilde{g}$. By the discussion there is a sequence $(\tilde{s}_n\colon n<\omega)$ such that for all n and $2j$ in the support of $\tilde{s}_n$, $\tilde{s}_n(2j)$ is a pleasant name, the empty condition forces that $\tilde{s}_n$ determines the value of $\tilde{g}(n)$ and for all n and $2j$ in the support of $\tilde{s}_n$, the empty condition (in $\mathbb{R}_{2j}$) forces that $\tilde{s}_{n+1}(2j)$

extends $\tilde{s}_n(2j)$. Then $\tilde{s}$ can be taken to be the coordinatewise union of the $\tilde{s}_n$.

Similarly, we can prove the following fact.

> Suppose $\tilde{g}$ is an $\mathbb{P}_\kappa$-name for a function from ω_1 to ω_1. Then there is a sequence $(\tilde{s}_\alpha\colon \alpha < \omega_1)$ of elements of $\tilde{S}_\kappa$, so that the empty condition in $\mathbb{R}_\kappa$ forces that the sequence is increasing and that $\tilde{s}_\alpha$ determines $\tilde{g}$ up to α.

With this fact in hand we can prove that all coseparable groups of cardinality $\aleph_1$ are free. Suppose A is a non-free coseparable group in the generic extension. By doing an initial segment of the forcing we can assume that A is in the ground model. By Lemma 2, we know that A is not coseparable in the generic extension by $\mathbb{R}_\kappa$. Let $h : A \to \mathbb{Z}/p\mathbb{Z}$ be a function which does not lift. By adding some of the Cohen reals to the ground model we can assume that h is in the ground model. Without loss of generality, we can assume that the set underlying A is ω_1. Suppose that $\tilde{g}$ is forced to be a lifting of h to $\mathbb{Z}$. Let $(\tilde{s}_\alpha\colon \alpha < \omega_1)$ be as above. Then if H is $\mathbb{R}_\kappa$-generic, H together with the sequence determines a function, φ from A to $\mathbb{Z}$. Since the value of any 3 elements is determined by some condition, this function must be a homomorphism which lifts h. But φ is in $V[H]$ which contradicts the choice of h.

3. $2^{\aleph_0}$-free may imply free

In [1] it is shown that if κ is a supercompact cardinal and κ Cohen reals are added to the universe then $2^{\aleph_0}$-free implies free. In order to prove Theorem 4, we need to know that if we begin with a supercompact cardinal κ and first add κ^+ Cohen subsets of ω_1 and then add κ subsets of ω, then $2^{\aleph_0}$-free implies free. The proof is relatively standard but we will present it below since we need the corollary.

LEMMA 17: *Suppose A is a non-free group. Then in any extension of the universe which preserves cofinalities and stationary sets, A is non-free.*

Proof: Consider such an extension and suppose that A is any non-free group. By restricting to a non-free subgroup of A we can assume that A is λ-free and the cardinality of A is λ or A is countable. By the singular compactness theorem λ is regular. The theorem is by induction on λ. By Pontryagin's criterion, being non-free is absolute upwards for countable groups. (More generally since satisfaction of sentences in $L_{\omega_1\omega}$ is absolute being non-free is absolute upwards for countable structures in any variety in a countable language.)

Suppose we know the result for all $\kappa < \lambda$ and A is λ-free, non-free and of cardinality λ. Then there is a λ-filtration $(A_\alpha\colon \alpha < \lambda)$ so that for a stationary set E, $A_{\alpha+1}/A_\alpha$ is not free for all $\alpha \in E$. By induction we know in the extension $A_{\alpha+1}/A_\alpha$ is not free. Hence A is not free in the extension. ∎

The lemma above applies in the more general situation of a variety in a countable language.

LEMMA 18: *Suppose that GCH is true in the ground model and let*

$$\mathbb{P} = \mathrm{Fn}(I, 2, \omega_1) \times \mathrm{Fn}(J, 2, \omega).$$

Suppose $\mathbb{Q}$ *is* $\mathrm{Fn}(\mu, 2, \omega_1) \times \mathrm{Fn}(\rho, 2, \omega)$ *where* $\mu, \rho > \aleph_1$. *If* G *is* $\mathbb{Q}$ *generic and* $K = V[G]$, *then in* K, $\mathbb{P}$ *is* $\aleph_2$*-c.c. and preserves stationary sets.*

Proof: It is easy to show that $\mathbb{P}$ is $\aleph_2$-c.c. in K, since $\mathbb{Q} \times \mathbb{P}$ is $\aleph_2$-c.c. over the ground model. But for amusement, we will give the proof only with the assumption that K is an extension by an $\aleph_2$-c.c. notion of forcing. Since K is an extension by an $\aleph_2$-c.c. poset, for any large enough regular κ there is a club C of subsets of $\mathrm{H}(\kappa)$ such that for all $N \in C$, G is generic over N. Suppose now that A is a maximal antichain in $\mathbb{P}$ (in K) and $\tilde{A}$ is a name for A. Choose $N \prec \mathrm{H}(\kappa)$ of cardinality $\aleph_1$ so that N is closed under sequences of length ω, $\tilde{A} \in N$ and G is generic over N. We will show that any element of $\mathbb{P}$ is compatible with an element of $N[G] \cap A$. Since $N[G]$ has cardinality $\aleph_1$, this suffices. Consider any element $(p_1, p_2) \in \mathbb{P}$. Since N is closed under countable sequences, $(p_1 \restriction (I \cap N), p_2 \restriction (J \cap N)) = q$ is in N. Hence q is compatible with some element r of $N[G] \cap A$. Finally r and (p_1, p_2) are compatible.

All that remains to see is that $\mathbb{P}$ doesn't destroy any stationary subsets of ω_1 (in K). Let S be any subset of ω_1 in K which is forced by the empty condition in $\mathbb{P}$ to be non-stationary. Suppose that $C \in K^{\mathbb{P}}$ is a club subset of ω_1 disjoint from S. Since $\mathbb{Q}$ is $\aleph_2$-c.c. we can choose a name $\tilde{S}$ for S which uses only $\mathrm{Fn}(L, 2, \omega_1) \times \mathrm{Fn}(T, 2, \omega)$ where the cardinality of L, T is $\aleph_1$. Moreover, we can have a name $\tilde{C}$ for C which uses only $\mathrm{Fn}(L, 2, \omega_1) \times \mathrm{Fn}(T, 2, \omega) \times \mathrm{Fn}(I', 2, \omega_1) \times \mathrm{Fn}(J', 2, \omega)$, for some $I' \subseteq I$, $J' \subseteq J$ of the size $\leq \aleph_1$. Since $\mu, \rho > \aleph_1$ we can think of $\tilde{C}$ as an $\mathrm{Fn}(\mu, 2, \omega_1) \times \mathrm{Fn}(\rho, 2, \omega)$-name. Then $\tilde{C}[G] \in K$ contradicts stationarity of S.
∎

THEOREM 19: *Suppose κ is a supercompact cardinal, V satisfies CH and $\mathbb{P} = \mathrm{Fn}(\mu, 2, \omega_1) \times \mathrm{Fn}(\rho, 2, \omega)$, where $\mu, \rho > \aleph_1$. Then $\mathbb{P}$ forces that every κ-free group is free.*

Proof: Suppose that G is $\mathbb{P}$-generic and A in $V[G]$ is κ-free. Let $\lambda = |A|$. Let $\tilde{A}$ be a name for A. Choose an embedding $j : V \to M$ so that $j(\kappa) > \lambda$ and M is closed under sequences of length $\max\{\kappa, \mu, \lambda, \rho\}$. Let H be a $j(\mathbb{P})$-generic set which contains $j"G$. Notice that A is isomorphic to the interpretation of $j"\tilde{A}$ in $M[j"G]$ and $M[H]$. (We denote this image as $j"A$.) If A is not free then $j"A$ is not free in $M[j"G]$ and hence by the two lemmas above also not in $M[H]$. However j extends to an elementary embedding of $V[G]$ into $M[H]$. So $j(A)$ is $j(\kappa)$-free and hence in $M[H]$, $j"A$ is free. ∎

COROLLARY 20: *If it is consistent with ZFC that a supercompact cardinal exists then both of the statements "every $2^{\aleph_0}$-free group is free and $2^{\aleph_0} < 2^{\aleph_1}$" and "every $2^{\aleph_0}$-free group is free and $2^{\aleph_0} = 2^{\aleph_1}$" are consistent with ZFC. Furthermore, if it is consistent that there is a supercompact cardinal then it is consistent that there is a cardinal $\kappa < 2^{\aleph_1}$ so that if κ Cohen reals are added to the universe then every κ-free group is free.*

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